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Approximations with modules having linear resolutions

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Abstract

Let Λ be a Koszul algebra over a field K. We study in this paper a class of modules closely related to the Koszul modules called weakly Koszul modules. It turns out that these modules have some special filtrations with modules having linear resolutions and therefore easy to describe minimal projective resolutions. We prove that if the Koszul dual of a finite-dimensional Koszul algebra is Noetherian then every finitely generated graded module has a weakly Koszul syzygy and as a consequence a rational Poincaré series. If Λ is selfinjective Koszul, we prove that the stable part of each connected component of the graded Auslander–Reiten quiver containing a weakly Koszul module is of the form $\mathbf{Z}A_\infty$, and if the Koszul dual of Λ is Noetherian, then every component has its stable part of the form $\mathbf{Z}A_\infty$.

Keywords: Koszul algebras; Weakly Koszul modules; Linear projective resolutions; Selfinjective algebras

1. Introduction

Let Λ be a Koszul algebra over a field K. We study in this paper a class of modules very closely related to the modules having linear resolutions over Λ, and we show that

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under certain conditions, every finitely generated graded Λ module can be approximated by modules having linear projective resolutions. We need to recall first some preliminary results and definitions.

By a graded algebra, we always mean an associative algebra over a field K, $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ where for each i, the K-dimension of Λ_i is finite, and where for each $i, j \geqslant 0$ we have $\Lambda_i \Lambda_j = \Lambda_{i+j}$. We also assume that the initial subalgebra $\Lambda_0 \simeq K \times \cdots \times K$. The graded radical of the algebra Λ is $J = \Lambda_1 \oplus \Lambda_2 \oplus \cdots$. Note that if our algebra Λ is finite-dimensional, then J is in fact, the Jacobson radical. By $\operatorname{mod} \Lambda$ we denote the category of all finitely generated Λ modules, by $\operatorname{Gr} \Lambda$ the subcategory of all graded modules and degree zero morphisms and by $\operatorname{gr} \Lambda$ the subcategory of all finitely generated graded Λ modules and degree zero homomorphisms. If $M = M_i \oplus M_{i+1} \oplus \cdots$ is a graded Λ module, since M_i is a Λ_0 summand of M/JM, we say that M_i is the "highest degree" part of M. For a module M in $\operatorname{mod} \Lambda$, we say that M is *quasi-Koszul* (*weakly Koszul*), if there exists a minimal projective resolution $(\mathcal{P})$ of M

$$\cdots \to \mathcal{P}_n \xrightarrow{f_n} \mathcal{P}_{n-1} \to \cdots \to \mathcal{P}_0 \xrightarrow{f_0} M \to 0, \tag{$\mathcal{P}$}$$

such that for each $i \geqslant 0$ we have $J^2\mathcal{P}_i \cap \operatorname{Ker} f_i = J \operatorname{Ker} f_i$ (for each $i, k \geqslant 0$, $J^{k+1}\mathcal{P}_i \cap \operatorname{Ker} f_i = J^k \operatorname{Ker} f_i$, respectively). Weakly Koszul modules were first introduced in [GM1] and were first called strongly quasi-Koszul. Note that the notions of quasi-Koszul and weakly Koszul make sense also in the nongraded case; we need only assume that the algebra is Noetherian and semiperfect with Jacobson radical J. If $M = M_i \oplus M_{i+1} \oplus \cdots$ is a graded Λ module then we say that M has a linear resolution if M is generated in degree i and if there exists a graded projective resolution of M

$$\cdots \to \mathcal{P}_n \xrightarrow{f_n} \mathcal{P}_{n-1} \to \cdots \to \mathcal{P}_0 \xrightarrow{f_0} M \to 0,$$

such that $\mathcal{P}_k$ is generated in degree $i + k$ for each $k \geqslant 0$. It is immediate that a linear projective resolution is always minimal in the sense that for each $i \geqslant 0$ we have $\operatorname{Ker} f_i \subset J\mathcal{P}_i$. It is clear that every weakly Koszul module is also quasi-Koszul and an easy induction argument shows that every graded module having a linear resolution is weakly Koszul. The Yoneda ext-algebra is the cohomology algebra

$$E(\Lambda) = \bigoplus_{n \geqslant 0} \operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$$

where the multiplication in $E(\Lambda)$ is defined by the Yoneda product, and we have a contravariant functor

$$E : \operatorname{mod} \Lambda \to \operatorname{Gr} E(\Lambda)$$

given by

$$E(M) = \bigoplus_{n \geqslant 0} \operatorname{Ext}^n_\Lambda(M, \Lambda_0)$$

with the obvious action of $E(\Lambda)$ on $E(M)$. The graded algebra Λ is a *Koszul* algebra if as a graded K-algebra, $E(\Lambda)$ is generated in degrees 0 and 1. This is equivalent to Λ_0 having a linear resolution in $\operatorname{gr}\Lambda$. We say that a graded Λ module is a *Koszul* module if it is generated in degree 0 and if it has a linear projective resolution. We also note [GM1], that a Λ module M is quasi-Koszul if and only if $E(M)$ is generated in degree zero as a graded $E(\Lambda)$ module.

The paper is organized as follows: in Section 2 we discuss some rather special filtrations of the weakly Koszul modules. We use these filtrations to show that a graded Λ-module is weakly Koszul if and only if its associated graded module has a linear projective resolution. In Section 3 we study in more detail relative extensions over a Koszul algebra and we prove that we can obtain the terms of a minimal projective resolution of a weakly Koszul module by adding up the corresponding terms of the minimal resolutions of certain modules with linear resolutions appearing in the special filtration of our module. Section 4 contains the main result of the paper. We first show that over a Koszul algebra a graded module is weakly Koszul if and only if its Koszul dual is a Koszul module over the Koszul dual of our algebra. Then we prove that if the Koszul dual of a finite-dimensional Koszul algebra is Noetherian, then each finitely generated graded module has a weakly Koszul syzygy. One application of this result deals with the rationality of Poincaré series. Recall that if Λ is a graded connected algebra and M is a finitely generated graded Λ module, then its Poincaré series $P_\Lambda^M(t)$ is defined as

$$P_\Lambda^M(t) = \sum_{n\geqslant 0} \dim_K \operatorname{Ext}_\Lambda^n(M, \Lambda_0)t^n.$$

It is well known that each finitely generated Koszul module has a rational Poincaré series if the connected algebra is a finite-dimensional Koszul algebra (and the same thing follows from Wilson's results (see [Wi,Za]) if Λ is any finite-dimensional, not necessarily connected Koszul algebra). On the other hand even in the commutative connected case the Poincaré series of a finitely generated module need not be rational (see [Ja], for instance). We prove in this section that if Λ is a finite-dimensional Koszul algebra whose Koszul dual $E(\Lambda)$ is Noetherian, then each finitely generated graded module has a rational Poincaré series. Section 5 is devoted to selfinjective finite-dimensional Koszul algebras. Ringel has shown in [R1] that if Λ is a selfinjective Koszul algebra of Loewy length greater than four, then every stable component of its Auslander–Reiten quiver containing a module with a linear resolution is a $\mathbf{Z}A_\infty$-component and, moreover, the modules having linear resolutions lie on the mouth of their components. We look in this section at the almost split sequences ending at weakly Koszul modules and show that they have at most two indecomposable summands in the middle term and that each component containing a weakly Koszul module contains a module whose almost split sequence has an indecomposable middle term therefore generalizing Ringel's result to components containing a weakly Koszul module. We then show that if the Koszul dual of the algebra is Noetherian, then the stable part of each component of the Auslander–Reiten quiver is of the form $\mathbf{Z}A_\infty$. Using this and a theorem of Bernstein, Gelfand, and Gelfand we obtain that the Auslander–Reiten components of the derived category of coherent sheaves over the projective space $\mathbf{P}^n$ for $n > 1$ are of the form $\mathbf{Z}A_\infty$.

2. Filtrations with modules having linear resolutions

We show in this section that over a Koszul algebra, every graded weakly Koszul module can be "filtered" by modules having linear projective resolutions. We use these results to show that a graded Λ module is weakly Koszul if and only if the associated graded module has a linear projective resolution. Related results have been independently obtained by Hashimoto [Ha]. We start with some preliminary results. The first two results are essentially proved in [GM1].

Lemma 2.1. *Let* $0 \to A \to B \to C \to 0$ *be exact in* $\operatorname{gr} \Lambda$. *The following are equivalent for each* $k \geqslant 0$:

(a) $A \cap J^k B = J^k A$;
(b) *The induced morphism* $\Lambda/J^k \otimes_\Lambda A \to \Lambda/J^k \otimes_\Lambda B$ *is a monomorphism.*

Proof. We have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \cap J^k B \arrow[r] \arrow[d] & J^k B \arrow[r] \arrow[d] & J^k C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A/A \cap J^k B \arrow[r] \arrow[d] & B/J^k B \arrow[r] \arrow[d] & C/J^k C \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The result follows now immediately. □

Corollary 2.2. *Let* $0 \to \Omega \to P \xrightarrow{f} M \to 0$ *be an exact sequence in* $\operatorname{gr} \Lambda$ *with* P *being a projective cover of* M. *The following are equivalent for each* $k \geqslant 0$:

(a) $\Omega \cap J^{k+1} P = J^k \Omega$;
(b) *The induced morphism* $\Omega/J^k \Omega \to JP/J^{k+1}P$ *is a monomorphism.*

Proof. There is an induced exact sequence $0 \to \Omega \to JP \to JM \to 0$. Now we can apply Lemma 2.1 to this sequence. □

It follows immediately that a Λ module M is weakly Koszul if and only if there exists a minimal projective resolution of M

$$\cdots \to \mathcal{P}_n \xrightarrow{f_n} \mathcal{P}_{n-1} \to \cdots \to \mathcal{P}_0 \xrightarrow{f_0} M \to 0,$$

such that for each $k \geqslant 0$, the functor $\Lambda/J^k \otimes_\Lambda$—takes the inclusions $\operatorname{Ker} f_i \to JP_i$ into monomorphisms. Note also that these results are also true in the nongraded case when Λ is semiperfect Noetherian with Jacobson radical J.

Lemma 2.3. *Let $0 \to A \to B \to C \to 0$ be exact in $\operatorname{gr}\Lambda$ and assume that for each $k \geqslant 0$ we have $J^k A = A \cap J^k B$ and that the generators in highest degree of B are in degree zero. Then $\langle A_0\rangle = A \cap \langle B_0\rangle$ where $\langle A_0\rangle$ is the homogeneous submodule of A generated by the degree 0 part of A, and similarly $\langle B_0\rangle$ is the submodule of B generated by the degree 0 part of B.*

Proof. It is trivial to show that the left-hand side is included in the right-hand side. To prove the equality it is enough to prove that for each $k \geqslant 0$ we have

$$\Lambda_k A_0 = A \cap \Lambda_k B_0$$

and it is clear that we need only show that the left-hand side contains the right one. Pick an arbitrary x in $A \cap \Lambda_k B_0 = A_k \cap \Lambda_k B_0$. But $\Lambda_k B_0$ is included in $J^k B$ so x is in $A_k \cap J^k A$. Since $J = \Lambda_1 \oplus \Lambda_2 \oplus \cdots$, a short degree analysis shows that x is in $\Lambda_k A_0$. $\square$

Theorem 2.4. *Let Λ be a Koszul algebra and let $M = M_0 \oplus M_1 \oplus \cdots$ be a graded weakly Koszul module with $M_0 \neq 0$. Let $K_M = \langle M_0\rangle$ be the submodule of M generated by the degree zero part. Then*:

(i) *K_M has a linear resolution.*
(ii) *$J^k M \cap K_M = J^k K_M$ for each $k \geqslant 0$.*
(iii) *M/K_M is weakly Koszul.*

Proof. (i) Let P_0 be the projective cover of M, and write

$$P_0 = P_0^0 \oplus P_0',$$

where P_0^0 is the summand of P_0 generated in degree 0 and P_0' is the summand of P_0 generated in positive degrees. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Omega K_M \arrow[r] \arrow[d] & P_0^0 \arrow[r] \arrow[d, hook] & K_M \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \Omega M \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

It is enough to prove that ΩK_M is the submodule of ΩM generated by the degree one part, since then we could use the fact that ΩM is again a weakly Koszul module and could proceed by induction. To show that ΩK_M equals the submodule of ΩM generated by the degree-one part, we first observe that we have an exact sequence

$$0 \to \Omega M \to JP_0 \to JM \to 0$$

and since M is weakly Koszul, we have by [GM1] that for each $k \geqslant 0$

$$J^k(\Omega M) = J^{k+1}P_0 \cap \Omega M = J^k(JP_0) \cap \Omega M.$$

We apply now the previous lemma with $A = \Omega M$ and $B = JP_0$. After shifting degrees we observe that

$$\langle(\Omega M)_1\rangle = \Omega M \cap \langle(JP_0)_1\rangle = \Omega M \cap \left(\left\langle\left(JP_0^0\right)_1\right\rangle \oplus \left\langle\left(JP_0'\right)_1\right\rangle\right).$$

But P_0' is generated in positive degrees, hence the degree-one part of JP_0' is 0. We have then

$$\langle(\Omega M)_1\rangle = \Omega M \cap \left\langle\left(JP_0^0\right)_1\right\rangle = \Omega M \cap \left\langle\left(P_0^0\right)_1\right\rangle = \Omega M \cap \left\langle\left(P_0^0\right)_0\right\rangle = \Omega M \cap P_0^0 = \Omega K_M.$$

(ii) We only need show $J^kM \cap K_M$ is included in J^kK_M. Let x be a nonzero homogeneous element of $J^kM \cap K_M$ of degree i. Since K_M is generated in degree 0, we get first that $i \geqslant k$, then we obtain that x is in J^iK_M but not in $J^{i+1}K_M$. It follows that $x \in J^kK_M$. Note that for this part we need not assume that the graded module M is weakly Koszul.

(iii) Follows from [GM1, 5.2] applied to the short exact sequence

$$0 \to K_M \to M \to M/K_M \to 0. \qquad \square$$

We use the previous theorem to prove that every weakly Koszul module can be filtered in a very nice way by "smaller" weakly Koszul modules. Let $M = M_0 \oplus M_1 \oplus \cdots$ be a weakly Koszul Λ module. We have the following pullback diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_M \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & K_L \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & K_M \arrow[r] & M \arrow[r] & L \arrow[r] & 0,
\end{tikzcd}$$

where K_L is the submodule of L generated by the highest degree part. We claim that for each $k \geqslant 0$ we have $J^kU \cap K_M = J^kK_M$ and $J^kM \cap U = J^kU$. This would imply in particular by applying [GM1, 5.3] to the top sequence that U is also a weakly Koszul module. To obtain these equalities we first tensor over Λ the above diagram with Λ/J^k. We get the following commutative diagram with exact rows:

$$\begin{tikzcd}
\Lambda/J^k \otimes K_M \arrow[r, "f"] \arrow[d, equal] & \Lambda/J^k \otimes U \arrow[r] \arrow[d, "\alpha"] & \Lambda/J^k \otimes K_L \arrow[r] \arrow[d, "\beta"] & 0 \\
\Lambda/J^k \otimes K_M \arrow[r, "f'"] & \Lambda/J^k \otimes M \arrow[r] & \Lambda/J^k \otimes L \arrow[r] & 0.
\end{tikzcd}$$

We know that f' and β are one-to-one by Lemma 2.1 and Theorem 2.4(ii). From the commutativity of the first square we see that f is also one-to-one, and then α is also one-to-one. It is also easy to see that $K_U = K_M$. Continuing, we obtain a sequence of graded Λ modules,

$$\langle M_0 \rangle = K_M = U_0 \subset U_1 \subset \cdots \subset U_p = M,$$

where for each $i \geqslant 0$, U_i is weakly Koszul, $K_{U_0} = K_{U_1} = \cdots = K_M$, $J^k M \cap U_i = J^k U_i$ for all $k \geqslant 0$, and where each quotient U_{i+1}/U_i has a linear projective resolution over Λ.

Another application of Theorem 2.4 explains the relationship between weakly Koszul and Koszul modules. We first recall that the *associated* graded module of a Λ module M is $G(M) = M/JM \oplus JM/J^2M \oplus \cdots$. The associated graded module of M is a graded module over the associated graded algebra $G(\Lambda)$. In our case, since Λ is Koszul, it is isomorphic as a graded algebra to its associated graded algebra so $G(M)$ can be viewed as a graded Λ module. We have the following:

Theorem 2.5. *Let Λ be a Koszul algebra and let M be a finitely generated graded Λ module. Then M is weakly Koszul if and only if $G(M)$ is a Koszul module.*

Proof. We start with a finitely generated graded Λ module M and assume that M is generated by a minimal set of homogeneous elements lying in degrees $k_0 < k_1 < \cdots < k_p$. Let K_M denote the submodule of M generated by the "highest" degree part, that is, $K_M = \langle M_{k_0} \rangle$. We have a short exact sequence of graded Λ modules $0 \to K_M \to M \to L \to 0$, and for each $k \geqslant 0$ we have $J^k K_M = K_M \cap J^k M$ by Theorem 2.4(ii). Furthermore, for each $k \geqslant 0$ we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & J^{k+1}K_M \arrow[r] \arrow[d, hook] & J^{k+1}M \arrow[r] \arrow[d, hook] & J^{k+1}L \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & J^k K_M \arrow[r] \arrow[d, hook] & J^k M \arrow[r] \arrow[d, hook] & J^k L \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & K_M \arrow[r] & M \arrow[r] & L \arrow[r] & 0.
\end{tikzcd}$$

By taking successive quotients, we obtain the following short exact sequence in $\operatorname{gr} G(\Lambda) = \operatorname{gr}(\Lambda)$:

$$0 \to G(K_M) \to G(M) \to G(L) \to 0. \tag{$*$}$$

We claim that the above sequence splits. To prove this claim, we may assume that $k_0 = 0$. For each j, let M'_{k_j} denote a Λ_0 complement in M_{k_j} of the degree k_j part of the submodule of M generated by the degree $0 = k_0, k_1, \dots, k_{j-1}$ parts. Let $M' = M'_{k_1} \oplus \cdots \oplus M'_{k_p}$. Then $M/JM = M_0 \oplus M'$ and it is easy to show we have $G(M) = G(K_M) + \langle M' \rangle$ where we also note that by looking at the degree zero part we get $G(M)_0 = G(K_M)_0 \oplus M'$. To show that this sum is direct, we show that $G(K_M) \cap \langle M' \rangle = 0$. Let $\hat{x}$ be a homogeneous element of degree i in the intersection $G(K_M) \cap \langle M' \rangle$. We can write $\hat{x} = \sum \hat{\lambda}\hat{y}$, where $\hat{\lambda} = \lambda + J^{i+1}$ with λ in Λ_i, and $\hat{y} = y + JK_M$ with y in $(K_M)_0$. Therefore, we have $\hat{x} = \sum \lambda y + J^{i+1}K_M$, with λ in Λ_i and y in $(K_M)_0$. On the other hand, since $\hat{x}$ is in $\langle M' \rangle$, we also have

$$\hat{x} = \sum \hat{\alpha}\hat{u} + \sum \hat{\beta}\hat{v} + \cdots,$$

where $\hat{\alpha} = \alpha + J^{i+1}$, $\hat{\beta} = \beta + J^{i+1}$, and so on, and $\alpha, \beta, \dots$ in Λ_i, and $\hat{u} = u + JM$ with u in M_{k_1}, $\hat{v} = v + JM$ with v in M_{k_2}, etc. It follows that in M we have $\sum \lambda y - (\sum \alpha u + \sum \beta v + \cdots)$ is in $J^{i+1}M$. But as a homogeneous element of M, $\sum \lambda y$ is in degree i, $\sum \alpha u$ is in degree $i + k_1$, etc. and then it follows easily that the above short exact sequence splits. We now use these observations to prove the theorem by induction on p. The case when $p = 0$ is clear since in that case M is generated in one degree and has a linear resolution. So we may assume that $p > 0$. From the split exact sequence $(*)$ we see that $G(M)$ is a Koszul module if and only if $G(L)$ and $G(K_M)$ are Koszul modules. Thus, if $G(M)$ is Koszul, we have by induction that both L and K_M are weakly Koszul, and the fact that M is also weakly Koszul follows from [GM1, 5.3]. For the reverse implication, if M is weakly Koszul, then K_M has a linear resolution and L is also weakly Koszul using [GM1] again, so by induction $G(L)$ and $G(K_M)$ are both Koszul therefore $G(M)$ is also a Koszul module. □

3. Relative extensions

It turns out that the extension $0 \to K_M \to M \to M/K_M \to 0$ from Theorem 2.4 is a particular case of a very interesting type of extension that will be studied in this section. We have the following definition.

Definition 3.1. Let Λ be a semiperfect Noetherian ring, or a graded ring with Jacobson radical J (graded radical, respectively, for the graded case). An extension of Λ modules (graded Λ modules, respectively)

$$0 \to A \to B \to C \to 0$$

is a called a *relative* extension if for each $k \geqslant 0$, we have $J^k A = J^k B \cap A$.

The following result is essentially proved in [GM1].

Proposition 3.2. *Let Λ be a Koszul algebra and let $0 \to A \to B \to C \to 0$ be a short exact sequence where $JA = JB \cap A$ (or a relative extension) where A is a quasi-Koszul (weakly Koszul, respectively) Λ-module. For each $n \geqslant 0$, let $\Omega^n(A)$, $\Omega^n(B)$, and $\Omega^n(C)$ denote the nth syzygies of A, B, and C, respectively. Then for each integer $n \geqslant 0$ we have a short exact sequence where $J\Omega^n(B) \cap \Omega^n(A) = J\Omega^n(A)$ (relative extension, respectively)*

$$0 \to \Omega^n(A) \to \Omega^n(B) \to \Omega^n(C) \to 0,$$

where $\Omega^n(A)$ is quasi-Koszul (weakly Koszul, respectively).

Proof. We may assume that B is not a projective Λ-module. If A is quasi-Koszul, it is clear that $\Omega(A)$ is again quasi-Koszul, and since $JA = A \cap JB$, we have an exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega(A) \arrow[r] \arrow[d] & \Omega(B) \arrow[r] \arrow[d] & \Omega(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{P}_A \arrow[r] \arrow[d] & \mathcal{P}_B \arrow[r] \arrow[d] & \mathcal{P}_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0, \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $\mathcal{P}_A$, $\mathcal{P}_B$, and $\mathcal{P}_C$ denote the projective covers of A, B, and C, respectively. We must show that the top sequence is a relative extension. Since the middle exact sequence splits, we have an induced commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega A \arrow[r] \arrow[d] & \Omega B \arrow[r] \arrow[d] & \Omega C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J\mathcal{P}_A \arrow[r] & J\mathcal{P}_B \arrow[r] & J\mathcal{P}_C \arrow[r] & 0.
\end{tikzcd}$$

Since A is quasi-Koszul, by tensoring with Λ/J, we obtain the following commutative exact diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & \Lambda/J \otimes \Omega A \arrow[r] \arrow[d] & \Lambda/J \otimes \Omega B \arrow[r] \arrow[d] & \Lambda/J \otimes \Omega C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda/J \otimes J\mathcal{P}_A \arrow[r] & \Lambda/J \otimes J\mathcal{P}_B \arrow[r] & \Lambda/J \otimes J\mathcal{P}_C \arrow[r] & 0.
\end{tikzcd}$$

The commutativity of the left square implies that the map $\Lambda/J \otimes \Omega A \to \Lambda/J \otimes \Omega B$ is one-to-one so that we obtain $J\Omega A = J\Omega B \cap \Omega A$ and the result follows by induction. The statement about weakly Koszul modules is proved in a similar manner. $\square$

We have the following immediate consequences.

Corollary 3.3. *Let Λ be a Koszul algebra and let $0 \to A \to B \to C \to 0$ be a relative extension where A is a quasi-Koszul Λ-module. For each $n \geqslant 0$, we have $\mathcal{P}_B^n = \mathcal{P}_A^n \oplus \mathcal{P}_C^n$, where $\mathcal{P}_A^n$, $\mathcal{P}_B^n$, and $\mathcal{P}_C^n$ denote the projective covers of $\Omega^n A$, $\Omega^n B$, and $\Omega^n C$, respectively. In particular,* $\operatorname{pd} B = \max(\operatorname{pd} A, \operatorname{pd} C)$.

It is easy to see now that if M is a weakly Koszul module, then a minimal projective resolution of M can be obtained directly from the minimal projective resolutions of modules having linear resolutions.

Proposition 3.4. *Let Λ be a Koszul algebra and let M be a weakly Koszul module and assume that M is generated in degrees $k_0 < k_1 < \cdots < k_p$. There exist graded Λ-modules $K_0, K_1, \ldots, K_p$ all having linear projective resolutions, such that* $\operatorname{pd} M = \max\{\operatorname{pd} K_i \mid i = 0, 1, \ldots, p\}$.

Proof. The proof follows from the above considerations and from Theorem 2.4 by induction on p, where we let $K_0 = K_M$. $\square$

An immediate consequence of this result and of [GMRSZ] is that over a finite-dimensional Koszul algebra, the finitistic dimension of the class of weakly Koszul modules is finite.

We end this section by mentioning some additional properties of the relative extensions. It is easy to see that if μ and η denote two equivalent extensions in $\operatorname{Ext}^1_\Lambda(C, A)$ then μ is a relative extension if and only if η is a relative extension. To see this assume that we have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r] \arrow[d, "h"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "f'"] & B' \arrow[r] & C \arrow[r] & 0. \end{tikzcd}$$

For each $k \geqslant 0$ the induced morphism $1 \otimes f : \Lambda/J^k \otimes A \to \Lambda/J^k \otimes B$ is a monomorphism if and only if the induced morphism $1 \otimes f' : \Lambda/J^k \otimes A \to \Lambda/J^k \otimes B'$ is a monomorphism. We also have the following result.

Proposition 3.5. *The class of relative extensions is closed under pullbacks and pushouts.*

Proof. (i) Let $0 \to A \xrightarrow{i} B \to C \to 0$ be exact with $J^k A = J^k B \cap A$ for each $k \geqslant 0$, and let $\gamma : A \to Y$ be a homomorphism. We have the following pushout diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\gamma"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r, "f"] & E \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

and, tensoring with Λ/J^k we get the following diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \Lambda/J^k \otimes A \arrow[r, "1\otimes i"] \arrow[d, "1\otimes\gamma"] & \Lambda/J^k \otimes B \arrow[r] \arrow[d, "h"] & \Lambda/J^k \otimes C \arrow[r] \arrow[d, equal] & 0 \\ & \Lambda/J^k \otimes Y \arrow[r, "1\otimes f"] & \Lambda/J^k \otimes E \arrow[r] & \Lambda/J^k \otimes C \arrow[r] & 0. \end{tikzcd}$$

We must show that $1 \otimes f$ is a monomorphism. We have a short exact sequence $0 \to A \xrightarrow{(i,-\gamma)} B \oplus Y \to E \to 0$, and since the induced map $1 \otimes i$ is a monomorphism, we see that the map $\Lambda/J^k \otimes A \xrightarrow{(1\otimes i,-1\otimes\gamma)} \Lambda/J^k \otimes (B \oplus Y)$ is also a monomorphism. This implies that the previous commutative diagram is a pushout diagram so the map $1 \otimes f$ is a monomorphism and we can apply Lemma 2.1 to see that the class of relative extensions is closed under pushouts.

(ii) Let $0 \to A \to B \to C \to 0$ be exact with $J^k A = J^k B \cap A$ for each $k \geqslant 0$ and let $\gamma : X \to C$ be a homomorphism. We have the following pullback diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & E \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

We must show that for each $k \geqslant 0$ the induced maps $\Lambda/J^k \otimes A \xrightarrow{1\otimes f} \Lambda/J^k \otimes E$ are all monomorphisms. This follows by looking at the leftmost square of the induced commutative digram

$$\begin{tikzcd}
 & \Lambda/J^k \otimes A \arrow[r, "1\otimes f"] \arrow[d, equal] & \Lambda/J^k \otimes E \arrow[r] \arrow[d] & \Lambda/J^k \otimes X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda/J^k \otimes A \arrow[r] & \Lambda/J^k \otimes B \arrow[r] & \Lambda/J^k \otimes C \arrow[r] & 0.
\end{tikzcd}$$

This completes the proof of the proposition. □

If Λ is a Koszul algebra, we define Δ_p to be the class of all the p-shifts of Koszul modules, that is Δ_p consists of the graded Λ-modules generated in degree p having linear projective resolutions. We have the following:

Proposition 3.6. *Let Λ be a Koszul algebra and let p and q be two integers with $p > q$. Then every relative extension $0 \to A \to B \to C \to 0$ where A is in Δ_p and C is in Δ_q, splits.*

Proof. Assume that we have a short exact sequence $0 \to A \to B \to C \to 0$ where A is in Δ_p and C is in Δ_q and for each $k \geqslant 0$, $J^k B \cap A = J^k A$. We must show that this sequence splits. Since $B = A + \langle B_q \rangle$, where $\langle B_q \rangle$ denotes the submodule of B generated by the degree q part, it is enough to prove that this sum is direct, so we must show that $A \cap \langle B_q \rangle = 0$. Let x be a homogeneous element of degree i, where $i \geqslant p > q$ in the intersection $A \cap \langle B_q \rangle = 0$. Then $x = \sum \lambda y$ where the λ's are in Λ_{i-q} and the y's are in B_q. So x is in $J^{i-q} B \cap A = J^{i-q} A$ and, since $A = \langle A_p \rangle$, we have $x = \sum \mu z$ where z is in A_p and the μ's are in J^t where $t \geqslant i - q$. We must also have $t + p = i$. But $t + p > t + q \geqslant i$ so we get a contradiction. □

Definition 3.7. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} \Lambda$. Its relative right perpendicular $\mathcal{Y} = \mathcal{X}_r^{\perp}$ is the full subcategory of $\operatorname{mod} \Lambda$ whose objects are those modules Y having the property that $\operatorname{Ext}_r^1(X, Y) = 0$ for all X in $\mathcal{X}$ where $\operatorname{Ext}_r^1(X, Y)$ denotes the class of all the relative extensions of Y by X.

Proposition 3.8. *$\mathcal{Y}$ is closed under relative extensions.*

Proof. Let $0 \to A \to B \to C \to 0$ be exact with A and C in $\mathcal{Y}$ and $J^k A = J^k B \cap A$ for all $k \geqslant 0$. To show that B is in $\mathcal{Y}$ we start with a short exact sequence $0 \to X \to E \to B \to 0$ with X in $\mathcal{X}$ and $J^k X = J^k E \cap X$ for all $k \geqslant 0$, and we must now show that this sequence splits. We have the following pullback diagram:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & E \arrow[r] & B \arrow[r] & 0.
\end{tikzcd}$$

By Proposition 3.6 the top sequence is a relative extension, so it splits since A is in $\mathcal{Y}$. We obtain the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & A \arrow[r, equal] \arrow[d] & A \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & F \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0, \\
 & & 0 & 0 &
\end{tikzcd}$$

where the morphism $A \to E$ is the composition of $M \to E$ with the right inverse of $A \to E$. Since $0 \to A \to B \to C \to 0$ is a relative extension, so is the vertical exact sequence ending in F since it is a pullback. We will show that the horizontal exact sequence ending at C is also a relative extension. Since C is in $\mathcal{Y}$ this will imply that this horizontal sequence splits and this will in turn imply that the horizontal sequence ending at B splits completing the proof. So we must show that for each $k \geqslant 0$ we have $J^k F \cap X = J^k X$. We tensor the entire diagram with Λ/J^k and keeping track of all the split exact sequences that we have so far, we obtain the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Lambda/J^k\otimes A \arrow[r, equal] \arrow[d] & \Lambda/J^k\otimes A \arrow[d] & \\
0 \arrow[r] & \Lambda/J^k\otimes X \arrow[r] \arrow[d, equal] & \Lambda/J^k\otimes E \arrow[r] \arrow[d] & \Lambda/J^k\otimes B \arrow[r] \arrow[d] & 0 \\
 & \Lambda/J^k\otimes X \arrow[r, "\varepsilon"] & \Lambda/J^k\otimes F \arrow[r] \arrow[d] & \Lambda/J^k\otimes C \arrow[r] \arrow[d] & 0. \\
 & & 0 & 0 &
\end{tikzcd}$$

It is easy to see using the "nine" lemma that the induced map ε is a monomorphism and this completes the proof. □

We recall from [AuS] that if $\mathcal{X}$ is a subcategory of a category $\mathcal{C}$, then by a right $\mathcal{X}$-approximation of an object M in $\mathcal{C}$, we mean a morphism $\pi: X_M \to M$, such that the induced morphism $\mathcal{C}(X, X_M) \to \mathcal{C}(X, M)$ is surjective for all objects X in $\mathcal{X}$. In a similar vein, if Λ is a Koszul algebra, we define a relative $\mathcal{X}$-approximation of a module M in $\operatorname{mod}\Lambda$ as a relative extension $0 \to K \to X_M \xrightarrow{\pi} M \to 0$ where the morphism $\pi: X_M \to M$ is a right $\mathcal{X}$-approximation.

Proposition 3.9. *Let $0 \to Y \to X \xrightarrow{\gamma} M \to 0$ be a relative extension with X in $\mathcal{X}$ and Y in $\mathcal{Y}$. Then γ is a right $\mathcal{X}$-approximation of M.*

Proof. Let X' be in $\mathcal{X}$ and let γ be a homomorphism $X' \to M$. We have the pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & E \arrow[r, "\varepsilon"] \arrow[d, "\beta"'] & X' \arrow[r] \arrow[d, "\gamma'"'] & 0 \\
0 \arrow[r] & Y \arrow[r] & X \arrow[r, "\gamma"] & M \arrow[r] & 0,
\end{tikzcd}$$

where the top sequence splits since Y is in $\mathcal{Y}$. Then if we compose β with the right inverse of ε we obtain a lifting of γ' to X. □

We end the section with the following result.

Proposition 3.10. *Let M be a Λ module and let $0 \to K \to X \xrightarrow{\pi} M \to 0$ be a relative extension with X in $\mathcal{X}$. Assume that we also have an exact sequence $0 \to K \to Y \to X' \to 0$ where X' is in $\mathcal{X}$ and Y is in $\mathcal{Y}$.*

(a) *If $\mathcal{X}$ is closed under extensions, then M has a relative right $\mathcal{X}$-approximation.*
(b) *If $\mathcal{X}$ is closed under relative extensions and if the sequence $0 \to K \to Y \to X' \to 0$ is also a relative extension, then M has a right $\mathcal{X}$-approximation that is also a relative extension.*

Proof. The proof of the proposition follows easily by applying our preceding results to the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & X \arrow[r, "\pi"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & M \arrow[r] & 0. \\
 & X' \arrow[r, equal] \arrow[d] & X' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

The second horizontal sequence is a relative extension since it is the pushout of a relative extension. In case (a), E belongs to $\mathcal{X}$ so the middle sequence is our relative right $\mathcal{X}$-approximation of M. For case (b), since the leftmost vertical sequence is a relative extension, so is its pushout and we obtain again that E is in $\mathcal{X}$, so we obtain again a relative right $\mathcal{X}$-approximation of M. $\square$

Note that one way to apply these result is in the case where $\mathcal{X}$ denotes the weakly Koszul modules since this class is always closed under relative extensions [GM1].

4. Weakly Koszul modules

We give in this section a new characterization of weakly Koszul modules, and we use this characterization to prove that over a finite-dimensional Koszul algebra whose Koszul dual is Noetherian every graded module has a weakly Koszul syzygy and we apply this to show the rationality of the Poincaré series of a finitely generated module. We start with the following result.

Lemma 4.1. *Let Λ be a Koszul algebra with Yoneda ext-algebra $E(\Lambda)$, and let M be a graded quasi-Koszul module such that $E(M)$ is in $\mathcal{K}_{E(\Lambda)}$ where E denotes the usual contravariant functor*

$$E : \operatorname{mod} \Lambda \to \operatorname{Gr} E(\Lambda)$$

and where $\mathcal{K}_\Lambda$ and $\mathcal{K}_{E(\Lambda)}$ denote the full subcategories of Koszul Λ-modules and Koszul $E(\Lambda)$-modules, respectively. Then, for each $k \geqslant 1$, $E(J^k M)$ is in $\mathcal{K}_{E(\Lambda)}$.

Proof. By induction it is enough to prove that $E(JM)$ is in $\mathcal{K}_{E(\Lambda)}$. We have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[r] & \Omega(M) \arrow[r] \arrow[d] & \Omega(M/JM) \arrow[r] \arrow[d] & JM \arrow[r] & 0 \\
 & & \mathcal{P}_M \arrow[r, equal] \arrow[d] & \mathcal{P}_{M/JM} \arrow[d] & & \\
0 \arrow[r] & JM \arrow[r] & M \arrow[r] \arrow[d] & M/JM \arrow[r] \arrow[d] & 0, & \\
 & & 0 & 0 & &
\end{tikzcd}$$

where $\mathcal{P}_M$ and $\mathcal{P}_{M/JM}$ denote the projective covers of M and of M/JM, respectively. Note also that since M is a quasi-Koszul module we have $J\Omega(M) = J^2\mathcal{P}_M \cap \Omega(M)$, and therefore we have $J\Omega(M/JM) \cap \Omega(M) = J^2\mathcal{P}_{M/JM} \cap \Omega(M/JM) \cap \Omega(M) = J^2\mathcal{P}_M \cap \Omega(M) = J\Omega(M)$. This means that the top row of this diagram satisfies the conditions of Proposition 3.2 and we obtain for each integer $k \geqslant 1$ a short exact sequence

$$0 \to \Omega^k(M) \to \Omega^k(M/JM) \to \Omega^{k-1}(JM) \to 0$$

and we also have for each $k \geqslant 1$ that $J\Omega^k(M) = J\Omega^k(M/JM) \cap \Omega^k(M)$. By applying the functor $\operatorname{Hom}_\Lambda(-, \Lambda_0)$ to this sequence we obtain for each integer $k \geqslant 1$ the following exact sequence:

$$0 \to \operatorname{Hom}_\Lambda\big(\Omega^{k-1}(JM), \Lambda_0\big) \to \operatorname{Hom}_\Lambda\big(\Omega^k(M/JM), \Lambda_0\big) \to \operatorname{Hom}_\Lambda\big(\Omega^k(M), \Lambda_0\big) \to 0.$$

Since Λ_0 is a semisimple module, we obtain for each integer $k \geqslant 1$ a short exact sequence

$$0 \to \operatorname{Ext}^{k-1}(JM, \Lambda_0) \to \operatorname{Ext}^k(M/JM, \Lambda_0) \to \operatorname{Ext}^k(M, \Lambda_0) \to 0$$

which by passing to the Koszul dual induces a short exact sequence of $E(\Lambda)$-modules

$$0 \to E(JM)(-1) \to E(M/JM) \to E(M) \to 0$$

and therefore $E(JM)(-1) = \Omega F(M)$ and we have that $E(JM)$ is in $\mathcal{K}_{E(\Lambda)}$. $\square$

Lemma 4.2. *Let Λ be a Koszul algebra and let $A \subseteq B$ be two quasi-Koszul modules with $JA = JB \cap A$ such that B/A is quasi-Koszul. Then $J^2A = J^2B \cap A$.*

Proof. Let $C = A/B$. We may clearly assume that B is not projective so by applying Proposition 3.2 to the exact sequence $0 \to A \to B \to C \to 0$, we obtain $J\Omega(A) = \Omega(A) \cap J\Omega(B)$. As before we have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega(A) \arrow[r] \arrow[d] & \Omega(B) \arrow[r] \arrow[d] & \Omega(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{P}_A \arrow[r] \arrow[d] & \mathcal{P}_B \arrow[r] \arrow[d] & \mathcal{P}_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and the induced commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega(A) \arrow[r] \arrow[d] & \Omega(B) \arrow[r] \arrow[d] & \Omega(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J\mathcal{P}_A \arrow[r] \arrow[d] & J\mathcal{P}_B \arrow[r] \arrow[d] & J\mathcal{P}_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & JA \arrow[r] \arrow[d] & JB \arrow[r] \arrow[d] & JC \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Tensoring this diagram with Λ/J we get the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Lambda/J\otimes\Omega(A) \arrow[r]\arrow[d] & \Lambda/J\otimes\Omega(B) \arrow[r]\arrow[d] & \Lambda/J\otimes\Omega(C) \arrow[r]\arrow[d] & 0 \\
0 \arrow[r] & \Lambda/J\otimes J\mathcal{P}_A \arrow[r]\arrow[d] & \Lambda/J\otimes J\mathcal{P}_B \arrow[r]\arrow[d] & \Lambda/J\otimes J\mathcal{P}_C \arrow[r]\arrow[d] & 0 \\
 & \Lambda/J\otimes JA \arrow[r]\arrow[d] & \Lambda/J\otimes JB \arrow[r]\arrow[d] & \Lambda/J\otimes JC \arrow[r]\arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The snake lemma implies that the leftmost map in the bottom sequence is a monomorphism and this is equivalent to having $J^2B\cap JA=J^2A$. But now we have $J^2B\cap A=J^2B\cap JB\cap A=J^2B\cap JA=J^2A$ and the proof of the lemma is complete. $\square$

We have the following characterization of weakly Koszul modules.

Theorem 4.3. *Let Λ be a Koszul algebra with Yoneda ext-algebra $E(\Lambda)$ and let $E:\operatorname{mod}\Lambda\to\operatorname{Gr}E(\Lambda)$ be the usual contravariant functor. Let M be a finitely generated Λ-module. Then M is weakly Koszul if and only if $E(M)$ is a Koszul $E(\Lambda)$-module.*

Proof. Assume that M is a weakly Koszul module. Then JM is again weakly Koszul [GM1]. We have seen in the proof of Lemma 4.1 that there is a short exact sequence of graded $E(\Lambda)$-modules

$$0\to E(JM)(-1)\to E(M/JM)\to E(M)\to 0.$$

Since JM is again weakly Koszul, we obtain in a similar fashion a short exact sequence of graded $E(\Lambda)$-modules

$$0\to E\bigl(J^2M\bigr)(-2)\to E\bigl(JM/J^2M\bigr)(-1)\to E(JM)(-1)\to 0.$$

By induction $E(M)$ has a linear resolution over $E(\Lambda)$ so $E(M)$ is a Koszul module over $E(\Lambda)$.

For the reverse implication, assume that $E(M)$ is a Koszul module over $E(\Lambda)$. Let $\mathcal{P}_0$ be the projective cover of M. We show first that for each $k\geqslant 1$ we have $J^k\Omega(M)=J^{k+1}\mathcal{P}_0\cap\Omega(M)$. We know from the proof of Lemma 4.1 that we have a short exact sequence

$$0\to\Omega(M)\to\Omega(M/JM)\to JM\to 0$$

and $J\Omega(M)=\Omega(M)\cap J\Omega(M/JM)$. Let k be maximal such that for all $1 \leqslant l \leqslant k$ we have

$$J^l\Omega(M) = \Omega(M) \cap J^l\Omega(M/JM). \tag{$*$}$$

We will show that $k = \infty$. From the above it follows that we have an exact sequence

$$0 \to J^{k-1}\Omega(M) \to J^{k-1}\Omega(M/JM) \to J^kM \to 0$$

and $J^k\Omega(M/JM) \cap J^{k-1}\Omega(M) = J^k\Omega(M/JM) \cap J^{k-1}\Omega(M) \cap \Omega(M)$, which using $(*)$ equals $J^k\Omega(M) \cap J^{k-1}\Omega(M) = J^k\Omega(M) = JJ^{k-1}\Omega(M)$. M/JM is weakly Koszul and so is $\Omega(M/JM)$ and then $J^{k-1}\Omega(M/JM)$ is also weakly Koszul. Now $E(\Omega(M)) = JE(M)$ and $E(M)$ being in $\mathcal{K}_{E(\Lambda)}$ imply that $JE(M)$ is in $\mathcal{K}_\Lambda$ so $E(\Omega(M))$ is in $\mathcal{K}_{E(\Lambda)}$ and by Lemma 4.1 $E(J^{k-1}\Omega(M))$ is in $\mathcal{K}_{E(\Lambda)}$ so that $J^{k-1}\Omega(M)$ is quasi-Koszul. We apply now Lemma 4.2 with $A = J^{k-1}\Omega(M)$ and $B = J^{k-1}\Omega(M/JM)$ and we obtain

$$J^2\bigl(J^{k-1}\Omega(M)\bigr) = J^{k-1}\Omega(M) \cap J^2J^{k-1}\Omega(M/JM).$$

We have $J^{k+1}\Omega(M) = J^{k-1}\Omega(M) \cap J^{k+1}\Omega(M/JM) \cap \Omega(M) = J^{k-1}\Omega(M/JM) \cap \Omega(M) \cap J^{k+1}\Omega(M/JM) = J^{k+1}\Omega(M/JM) \cap \Omega(M)$. This contradicts the maximality of k so $k = \infty$. Therefore for each nonnegative integer k we have

$$J^k\Omega(M) = J^k\Omega(M/JM) \cap \Omega(M). \tag{$**$}$$

But $\mathcal{P}_0$ is also the projective cover of M/JM and $M/JM)$ has a linear resolution so $J^k\Omega(M/JM) = J^{k+1}\mathcal{P}_0 \cap \Omega(M)$ for each $k \geqslant 1$. It now follows using $(**)$ and the above that for each $k \geqslant 1$ we have $J^k\Omega(M) = J^{k+1}\mathcal{P}_0 \cap \Omega(M)$. The result follows now by induction since for each $i \geqslant 1$ we have short exact sequences

$$0 \to \Omega^i(M) \to \Omega^i(M/JM) \to \Omega^{i-1}(JM) \to 0$$

and $J\Omega^i(M) = \Omega^i(M) \cap J\Omega^i(M/JM)$. □

Proposition 4.4. *Let Γ be a Koszul algebra and let M be a finitely generated graded Γ-module having a finitely generated finite projective resolution. Then, there exists an integer k such that $M_{\geqslant k}$ has a linear resolution over Γ and $M/M_{\geqslant k}$ is finite-dimensional, where $M_{\geqslant k} = M_k \oplus M_{k+1} \oplus \cdots$.*

Proof. If M is finite-dimensional, say $M = M_{i_0} \oplus M_{i_1} \oplus \cdots \oplus M_{i_n}$, we can choose $k = i_n$ since $M_{\geqslant i_n}$ has a linear resolution being a semisimple Γ-module and $M/M_{\geqslant i_n}$ is finite-dimensional. If M is infinite-dimensional, look at its projective resolution

$$\cdots \to \mathcal{P}_n \to \mathcal{P}_{n-1} \to \cdots \to \mathcal{P}_0 \to M \to 0. \tag{$*$}$$

We have an induced long exact sequence

$$\cdots \to (P_n)_{\geqslant k} \to (P_{n-1})_{\geqslant k} \to \cdots \to (\mathcal{P}_0)_{\geqslant k} \to M_{\geqslant k} \to 0, \tag{$**$}$$

where k is the largest degree in which a summand of $\mathcal{P}_n$ can be generated. The proposition follows now by induction from the fact that if we have a short exact sequence of graded modules generated in degree k

$$0 \to A \to B \to C \to 0$$

with A and B having linear resolutions and $JA = JB \cap A$, then C also has a linear resolution. $\square$

We are ready now to prove the main result of this section.

Theorem 4.5. *Let Λ be a finite-dimensional Koszul algebra such that its Yoneda ext-algebra $E(\Lambda)$ is Noetherian, and let M be a finitely generated Λ-module. Then there exists a nonnegative integer k having the property that $\Omega^k(M)$ is weakly Koszul.*

Proof. Observe first that since Λ is finite-dimensional, $E(\Lambda)$ has finite global dimension. Let $E : \operatorname{mod}\Lambda \to \operatorname{Gr} E(\Lambda)$ be the functor $E = \operatorname{Ext}^*_\Lambda(-, \Lambda_0)$. We prove first that $E(M)$ is a finitely generated graded $E(\Lambda)$-module. We do this by induction on the Loewy length of M. The statement is obvious if M is semisimple. For the induction step, look at the exact sequence $0 \to JM \to M \to M/JM \to 0$. We have then an induced exact sequence of graded $E(\Lambda)$-modules

$$E(M/JM) \xrightarrow{\alpha} E(M) \xrightarrow{\beta} E(JM).$$

We have an induced exact sequence $0 \to \operatorname{Im}(\alpha) \to E(M) \to \operatorname{Im}(\beta) \to 0$. Since $E(\Lambda)$ is Noetherian, our induction shows that $\operatorname{Im}(\alpha)$ and $\operatorname{Im}(\beta)$ are finitely generated so $E(M)$ is finitely generated. We now apply the previous theorem to $E(M)$: for some $k \geqslant 0$ we have that $E(M)_{\geqslant k}$ has a linear resolution. But

$$E(M)_{\geqslant k} = \bigoplus_{j \geqslant k} \operatorname{Ext}^j_\Lambda(M, \Lambda_0)(-k) = \bigoplus_{j \geqslant 0} \operatorname{Ext}^j_\Lambda\big(\Omega^k(M), \Lambda_0\big) = E\big(\Omega^k(M)\big).$$

This implies that $E(\Omega^k(M))$ has a linear projective resolution over $E(\Lambda)$ so $\Omega^k(M)$ is weakly Koszul by Theorem 4.3. $\square$

We will apply now our results to Poincaré series of finitely generated graded modules. We start with the following lemma.

Lemma 4.6. *Let Λ be a finite-dimensional Koszul algebra and let M be a finitely generated graded weakly Koszul module. Then the Poincaré series*

$$P^M_\Lambda(t) = \sum_{m \geqslant 0} \dim_K \operatorname{Ext}^m_\Lambda(M, \Lambda_0) t^m$$

of M is rational.

Proof. We do induction on p where M is generated in degrees $i_0 < i_1 < \cdots < i_p$. We must show first that the Poincaré series of a Koszul module is rational. If Λ is a connected Koszul algebra we can use the well-known formula

$$P_\Lambda^M(t) = H_M(-t)/H_\Lambda(-t),$$

where $H_M(t) = \sum_{n\geqslant 0}(\dim_K M_n)t^n$ denotes the Hilbert series of a Koszul module $M = M_0 \oplus M_1 \oplus \cdots$, so if Λ and M are finite-dimensional, then $H_M(-t)$ and $H_\Lambda(-t)$ are polynomials and so $P_\Lambda^M(t)$ is rational. For the general nonconnected case we do not have such a nice formula but the rationality still follows from Wilson's paper [Wi] (see also [GMRSZ,Za]). For the induction step, we have by Theorem 2.4 the exact sequence

$$0 \to K_M \to M \to M/K_M \to 0$$

and by Corollary 3.3 we have that the Poincaré series of M can be obtained by adding the Poincaré series of K_M and of M/K_M and the lemma follows by induction. □

Theorem 4.7. *Let Λ be a finite-dimensional Koszul algebra whose Koszul dual $E(\Lambda)$ is Noetherian and let M be a finitely generated graded Λ module. Then the Poincaré series $P_\Lambda^M(t)$ of M is rational.*

Proof. We know that $\Omega^n M$ is weakly Koszul for some $n > 0$ hence $P_\Lambda^{\Omega^n M}(t)$ is rational by Lemma 4.6. On the other hand

$$P_\Lambda^M(t) = t^n P_\Lambda^{\Omega^n M}(t) + Q(t),$$

where $Q(t)$ is a polynomial in t hence the result follows. □

For other cases in which the Poincaré series of each finitely generated module is rational, see [Av].

5. Selfinjective Koszul algebras

In this section we study finite-dimensional selfinjective Koszul algebras. We prove that over such an algebra, the middle term of the almost split sequence ending at a weakly Koszul module has at most two indecomposable summands and we prove that if Λ is a selfinjective Koszul algebra of Loewy length greater than three whose Koszul dual is Noetherian, then every graded stable component of its Auslander–Reiten quiver is of the form $\mathbf{Z}A_\infty$. Such Koszul algebras include Yoneda ext-algebras of skew group algebras of polynomial algebras (for instance, exterior algebras if the group is trivial) or preprojective algebras of Euclidean quivers, and they have also appeared in the study of Artin–Shelter regular algebras (see [AS,GTM,M1,M2,Sm]).

We shall need the following facts about graded Λ-modules (see also [M1,Sm]). Let P be an indecomposable projective module generated in degree i. Then $P^* = \mathrm{Hom}_\Lambda(P, \Lambda)$ is an indecomposable projective Λ^{op}-module generated in degree $-i$. We also know that if M is a graded Λ-module, then its dual $DM = \mathrm{Hom}_K(M, K)$ is a graded Λ^{op}-module by putting $(DM)_i = D(M_{-i})$ for each i. Since all the indecomposable projective modules have the same Loewy length, if Λ has Loewy length n and P is an indecomposable projective module generated in degree i then P^* is generated in degree $-i$ and its socle is in degree $-i+n-1$ so DP^* is generated in degree $i-n+1$. Assume now that M has a linear resolution $(\mathcal{P})$

$$\cdots \to \mathcal{P}_n \xrightarrow{f_n} \mathcal{P}_{n-1} \to \cdots \to \mathcal{P}_0 \xrightarrow{f_0} M \to 0 \qquad (\mathcal{P})$$

and that M is generated in degree i. Applying $\mathrm{Hom}_\Lambda(-, \Lambda)$ and then the duality D to this resolution we get since Λ is selfinjective the following projective resolution of $D\,\mathrm{Hom}_\Lambda(M, \Lambda)$:

$$\cdots \to D\mathcal{P}_n^* \to D\mathcal{P}_{n-1}^* \to \cdots \to D\mathcal{P}_0^* \to D\,\mathrm{Hom}_\Lambda(M, \Lambda) \to 0,$$

where for each k, the projective module $\mathcal{D}P_k^*$ is generated in degree $i+k-n+1$ and therefore νM is again a module having a linear resolution where $\nu = D\,\mathrm{Hom}_\Lambda(-, \Lambda)$ is the Nakayama equivalence functor. Using the exact sequence

$$0 \to \tau M \to D\mathcal{P}_1^* \to D\mathcal{P}_0^* \to \nu M \to 0,$$

we see that if M is a module generated in degree i and having a linear resolution, then its Auslander–Reiten translate $\tau M = D\,\mathrm{Tr}\,M$ has a linear resolution and is generated in degree $i-n+3$.

We will need the following result.

Proposition 5.1. *Let Λ be a finite-dimensional selfinjective Koszul algebra and let M be a weakly Koszul module. Then τM is also weakly Koszul.*

Proof. For the proof, observe first that over a selfinjective algebra, the Nakayama equivalence ν takes radicals of modules into radicals of modules, preserves intersections and takes projective modules into projective modules. It follows from the definition that if M is weakly Koszul, the so is $\nu(M)$. Finally, we also know that the second syzygy of a weakly Koszul module is weakly Koszul, and therefore $\tau M = \Omega^2 \nu(M)$ is again weakly Koszul and the proof is complete. □

Using this result and the previous remarks we see that if M is a weakly Koszul module generated in degrees $i_0 < i_1 < \cdots < i_p$, then τM is a weakly Koszul modules generated in degrees $i_0 - n + 3 < i_1 - n + 3 < \cdots < i_p - n + 3$ where n denotes the Loewy length of Λ.

An immediate consequence of Theorem 4.5 is that over a finite-dimensional selfinjective Koszul algebra whose ext-algebra is Noetherian, every component of the Auslander–Reiten quiver contains weakly Koszul modules.

Theorem 5.2. *Let Λ be a finite-dimensional selfinjective Koszul algebra such that its Yoneda ext-algebra is Noetherian. Let M be a graded Λ module. Then $\tau^n M$ is weakly Koszul for some $n \geqslant 1$ where τ denotes the Auslander–Reiten translation.*

We turn our attention to the study of the almost split sequences ending at the weakly Koszul modules. We will need the following result.

Lemma 5.3. *Let Λ be a selfinjective Koszul algebra and let A be an indecomposable nonprojective weakly Koszul module and let*

$$0 \to \tau A \xrightarrow{g} B_1 \oplus B_2 \oplus \cdots \oplus B_k \xrightarrow{f} A \to 0$$

be the almost split sequence ending at A where $f = [\, f_1 \quad f_2 \quad \ldots \quad f_k \,]$ and $g = [\, g_1 \quad g_2 \quad \ldots \quad g_k \,]^{\mathrm{T}}$. Then exactly one of the f_i is an epimorphism, and if say f_1 is an epimorphism, then for each $i \neq 1$, g_i is an epimorphism and g_1 is a monomorphism.

Proof. There is nothing to prove if $k = 1$ so we may assume that $k > 1$. We show first that at most one of the f_i is an epimorphism. If $f_1 : B_1 \to A$ and $f_2 : B_2 \to A$ are both epimorphisms then the homomorphism $f' : B' = B_2 \oplus \cdots \oplus B_k \to A$ where $f' = [\, f_2 \quad f_3 \quad \ldots \quad f_k \,]$ is also an epimorphism. It now follows immediately that $g_1 : \tau A \to B_1$ is an epimorphism and therefore we obtain an epimorphism from τA onto A which is impossible since by our earlier remarks the degrees containing a set of minimal homogeneous generators of τA are obtained from the degrees containing a set of minimal homogeneous generators of A by an upward shift and therefore not all the minimal generators of A can be covered by elements of τA. Hence each morphism $f_i : B_i \to A$ is a monomorphism for all $i > 1$ and g_1 is a monomorphism. If g_i is a monomorphism for some $i > 1$, then we have a monomorphism $\tau A \to A$ but this cannot happen either by a similar degree analysis. Let us assume now that none of the f_i is an epimorphism. This means that each of the f_i is a monomorphism, and therefore if $B = B_1 \oplus B_2 \oplus \cdots \oplus B_k$ is generated in degrees $j_0 < \cdots < j_s$, and if A is generated in degrees $i_0 < \cdots < i_p$, we must have $i_p \leqslant j_s$. On the other hand $\tau A \subset B$ and this implies that $k_t \geqslant i_p$ where τA is generated in degrees $k_0 < \cdots < k_t$ contradicting our earlier remarks. □

Lemma 5.4. *Let $f : B \to A$ be an irreducible epimorphism and let A be a weakly Koszul module. Then* Ker f *is not simple.*

Proof. If the kernel of f is simple then A and B have their minimal homogeneous generators in the same degrees since Ker f is contained in the radical of B. Let

$$0 \to \tau A \to B \oplus B' \xrightarrow{h} A \to 0$$

be the almost split sequence ending at A, where $h = [\, f \quad f' \,]$ and f' is an irreducible monomorphism by the previous lemma. Just as in the above proof we get the contradiction $k_t \geqslant i_p$ where τA is generated in degrees $k_0 < \cdots < k_t$ and A is generated in degrees $i_0 < \cdots < i_p$. □

We have the following easy consequence.

Corollary 5.5. *Let $f : B \to A$ be an irreducible epimorphism and let A be a weakly Koszul module. Then the induced morphism $\tau f : \tau B \to \tau A$ is again an irreducible epimorphism.*

Proof. We have a short exact sequence $0 \to K \to B \xrightarrow{f} A \to 0$ where K is the kernel of f. Since K is not simple and f is irreducible, we have $JK = JB \cap K$ so that we get a commutative diagram with exact rows and columns where $\mathcal{P}_K$, $\mathcal{P}_B$, and $\mathcal{P}_A$ denote the projective covers of K, B, and A, respectively,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega(K) \arrow[r] \arrow[d] & \Omega(B) \arrow[r] \arrow[d] & \Omega(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{P}_K \arrow[r] \arrow[d] & \mathcal{P}_B \arrow[r] \arrow[d] & \mathcal{P}_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

In addition, the induced morphism $\Omega B \to \Omega A$ is again an irreducible map, and since ΩA is again weakly Koszul we can repeat the argument to obtain that the induced map $\Omega^2 B \to \Omega^2 A$ is an irreducible epimorphism. Since over a selfinjective algebra $\tau = \nu\Omega^2$ the result follows. □

It is easy to see that as a consequence of the above discussion we obtain that over a selfinjective Koszul algebra, every predecessor in the Auslander–Reiten quiver of a weakly Koszul module is again weakly Koszul. Let now A be an indecomposable nonprojective weakly Koszul module lying in a connected component of the Auslander–Reiten quiver of Λ and let

$$0 \to \tau A \to B_1 \oplus B_2 \oplus \cdots \oplus B_k \xrightarrow{f} A \to 0$$

be the almost split sequence ending at A where $f = [\, f_1 \quad f_2 \quad \ldots \quad f_k \,]$ and $k > 1$. We may assume that $f_1 : B_1 \to A$ is an irreducible epimorphism and that f_i are irreducible monomorphisms for $i \neq 1$. Using the preceding results we have an almost split sequence

$$0 \to \tau^2 A \to \tau B_1 \oplus \tau B_2 \oplus \cdots \oplus \tau B_k \xrightarrow{\tau f} \tau A \to 0$$

and the morphism $\tau f_1 : \tau B_1 \to \tau A$ is an epimorphism and $\tau f_i : \tau B_i \to \tau A$ are monomorphisms for each $i \neq 1$. We can now prove:

Theorem 5.6. *Let Λ be a selfinjective Koszul algebra of Loewy length greater than three and let $\mathcal{C}$ be a connected component of its Auslander–Reiten quiver containing a nonprojective graded indecomposable weakly Koszul module A. Let $\mathcal{C}_A$ denote the cone of A that is the full subquiver of $\mathcal{C}$ consisting of all the predecessors of A. Then each module M in $\mathcal{C}_A$ has the property that the middle term of the almost split sequence ending at M has at most two indecomposable summands.*

Proof. Since the Loewy length is greater than two, the middle term of the almost split sequence ending at a weakly Koszul module cannot have a projective–injective summand since an injective module modulo its socle is not weakly Koszul. Since for each M in the cone of A, the cone of M is part of the cone of A it is enough to prove the result for A. By the preceding results we may assume that we have an almost split sequence

$$0 \to \tau A \xrightarrow{g} B_1 \oplus B_2 \oplus B_3 \oplus K \xrightarrow{f} A \to 0,$$

where $g = [\,g_1 \quad g_2 \quad g_3 \quad g_4\,]^{\mathrm{T}}$ and $f = [\,f_1 \quad f_2 \quad f_3 \quad f_4\,]$, where B_1, B_2, B_3 are indecomposable, $f_1 : B_1 \to A$ is an epimorphism, all the other f_i are irreducible monomorphisms and by shifting the sequence τf_1 and $\tau^{-1} f_1$ are again irreducible epimorphisms. Considering also the almost split sequence ending at $\tau^{-1} A$

$$0 \to A \to \tau^{-1} B_1 \oplus \tau^{-1} B_2 \oplus \tau^{-1} B_3 \oplus \tau^{-1} K \to \tau^{-1} A \to 0$$

and if we denote by $l(X)$ the length of a module X, we obtain

$$\sum_1^3 l(B_i) + \sum_1^3 l\bigl(\tau^{-1} B_i\bigr) + l(K) + l\bigl(\tau^{-1} K\bigr) = 2l(A) + l(\tau A) + l\bigl(\tau^{-1} A\bigr).$$

From the existence of almost split sequences $0 \to B_2 \to A \oplus C \to \tau^{-1} B_2 \to 0$ and $0 \to B_3 \to A \oplus C' \to \tau^{-1} B_3 \to 0$, we obtain the relations $l(B_2) + l(\tau^{-1} B_2) = l(A) + l(C)$ and $l(B_3) + l(\tau^{-1} B_3) = l(A) + l(C')$ and we then get the identity

$$\begin{aligned} &l(B_1) + l\bigl(\tau^{-1} B_1\bigr) + 2l(A) + l(C) + l\bigl(C'\bigr) + l(K) + l\bigl(\tau^{-1} K\bigr) \\ &\quad = 2l(A) + l(\tau A) + l\bigl(\tau^{-1} A\bigr). \end{aligned}$$

It follows that:

$$l(B_1) + l\bigl(\tau^{-1} B_1\bigr) \leqslant l(\tau A) + l\bigl(\tau^{-1} A\bigr).$$

On the other hand f_1, τf_1 and $\tau^{-1} f_1$ are epimorphisms and these facts imply $l(\tau^{-1} A) < l(\tau^{-1} B_1)$ and $l(\tau A) < l(\tau B_1)$ hence $l(\tau^{-1} A) + l(\tau A) < l(\tau^{-1} B_1) + l(\tau B_1)$, obtaining a contradiction and completing the proof of the theorem. $\square$

The main results of the section follow immediately.

Theorem 5.7. *Let Λ be a selfinjective Koszul algebra of Loewy length greater than three and let $\mathcal{C}$ be a graded component of the Auslander–Reiten quiver containing a weakly Koszul module. Then the stable part of $\mathcal{C}$ is of the form $\mathbf{Z}A_\infty$.*

Proof. Let $\mathcal{C}$ be a graded connected component and let M be a weakly Koszul module in $\mathcal{C}$. By taking shifts of the Auslander–Reiten translation it follows from the previous result that for each module in $\mathcal{C}$ the middle term of the almost split sequence ending at the module has at most two indecomposable summands. To exclude the $\mathbf{Z}A_\infty^\infty$ case, we must show that $\mathcal{C}$ contains a module for which the middle term of the almost split sequence is indecomposable. Let A be a weakly Koszul module in $\mathcal{C}$ of smallest length, so each irreducible morphism with target A is an epimorphism. (Note that A cannot be projective since the radical of a projective module is again Koszul so it is also weakly Koszul.) The result follows now immediately from Lemma 5.3 and from Theorem 5.6. □

Theorem 5.8. *Let Λ be a selfinjective Koszul algebra of Loewy length greater than three such that its Yoneda ext-algebra $E(\Lambda)$ is Noetherian. Then the stable part of each graded component of the Auslander–Reiten quiver is of the form $\mathbf{Z}A_\infty$.*

Proof. Let $\mathcal{C}$ be a graded stable component and let M be a module in $\mathcal{C}$. Theorem 5.2 implies that $\tau^n M$ is weakly Koszul for some positive integer n, so we may apply the previous theorem. □

Note that the above theorem may fail if we do not assume that the Loewy length of the algebra is greater than three. For instance, if Λ is the exterior algebra of a 2-dimensional vector space, then the Loewy length of Λ is three and the Auslander–Reiten components are stable tubes and components that are not of the type $\mathbf{Z}A_\infty$.

We can apply now the previous theorem to the case where Λ is the exterior algebra of an $(n+1)$-dimensional vector space. Λ is then a selfinjective Koszul algebra whose Koszul dual is the commutative polynomial algebra in $n+1$ variables so it is Noetherian. Bernstein, Gelfand, and Gelfand proved in [BGG] that the derived category $D^b(\operatorname{coh}\mathbf{P}^n)$ of coherent sheaves over the projective space is equivalent to the stable graded module category $\underline{\operatorname{gr}\Lambda}$ over the exterior algebra Λ. Using this result and the previous theorem we obtain the following:

Theorem 5.9. *Let $D^b(\operatorname{coh}\mathbf{P}^n)$ denote the derived category of coherent sheaves over the projective space $\mathbf{P}^n$ for $n > 1$. Then each component of its Auslander–Reiten quiver is of the form $\mathbf{Z}A_\infty$.*

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